Coupled Strip Transmission Line With Three Center Conductors

S. YAMAMOTO, STUDENT MEMBER, IEEE, T. AZAKAMI, MEMBER, IEEE, AND K. ITAKURA

Abstract—**An exact analysis is made for the shielded coupled strip transmission line with three center conductors. By means of the immittance matrices presented, the electrical behavior of the coupled strip transmission line of this type may be completely described. Design formulas which enable one to evaluate the cross-section dimensions from the desired values of characteristic immittances are derived for the two kinds of the line configurations. Equivalent circuits of the two-port networks for various port conditions are also presented. In addition, an application is discussed involving the asymmetrical coupled strip transmission line with two center conductors. The Appendix gives the derivation of the design formulas by means of conformal mapping techniques.**

I. Introduction

THE multiconductor transmission line has been found to have interesting properties and, consequently, has received much attention [1]. Applications have been made to antennas [2], transmission lines for high speed computers [3], and various circuit components [4]–[8]. Under the appropriate boundary conditions, multiconductor transmission lines have been utilized as filters in the microwave region. Comb-line and interdigital filters are two examples. The former has been treated by Matthaei [5] and the latter has been investigated by Matthaei [4] and more recently by Wenzel [8].

In these papers, the design data have been obtained under the assumption that no direct coupling exists between nonadjacent conductors. Neglect of this type of coupling allows TEM propagation along multiconductor transmission lines to be described in terms of only two independent modes, and therefore simplifies the analysis. In so far as rectangular bars or circular cylindrical rods are used as component lines, this assumption is considered to be accurate. However, in a more precise analysis or when the other configurations are employed, it is necessary to take into account the coupling between nonadjacent lines as is done in this paper.

From the viewpoint of the exact synthesis of distributed TEM-mode networks [9], [10], it is important and desirable to introduce the coupled quarter-wave transmission lines with more than three center conductors as the canonical sections. No exact theory, however, seems to be available for such coupled transmission lines, including design formulas for the cross section of the line configurations as well as immittance matrices.

The purpose of this paper is to present the results of an exact analysis for the coupled three-conductor transmission line with common return in strip-line form, in which significant coupling exists between nonadjacent lines. The procedures presented will be applicable to more complicated coupled transmission lines.

Manuscript received March 1, 1966; revised July 7, 1966.
S. Yamamoto and K. Itakura are with the Department of Electrical Communication Engineering, School of Engineering, Osaka University, Osaka, Japan.
T. Azakami is with the Division of Electrical Engineering, Nara Technical College, Nara, Japan.

II. Fundamental Modes

We consider the lossless coupled transmission line consisting of three parallel conductors above ground with uniform cross section, infinitely long, and symmetrical about the central conductor as shown in Fig. 1 where the circles represent the coupled conductors. The medium surrounding the conductors is assumed to be homogeneous and isotropic. Then it is possible to describe TEM propagation along such a structure in terms of three orthogonal modes, that is, field distribution at any transverse plane can be expressed as a linear combination of these three fundamental TEM-modes [11]. Of course, they have a unique propagation constant, but have different characteristic immittances. Proper choice of the fundamental modes is necessary so as to simplify the analysis.

A. Fundamental Modes for the Derivation of Impedance Matrix

Figure 2 shows a set of fundamental modes which are given in convenient forms for deriving the impedance matrix of the coupled transmission line six-port and are designated as A-, B-, and C-mode.

Now consider a system of three conductors and ground as shown in Fig. 1. The potentials of the conductors to ground are related to the charges on them per unit length by

$$\begin{bmatrix} V_{I} \\ V_{II} \\ V_{III} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{12} \\ p_{13} & p_{12} & p_{11} \end{bmatrix} \cdot \begin{bmatrix} Q_{I} \\ Q_{II} \\ Q_{III} \end{bmatrix} \qquad (1)$$

where the p's are known as the coefficients of potential.

The static capacitances of conductors to ground for the fundamental modes are given by

$$C_{IA} = C_{IIIA} = (Q_I/V_I)_{Q_I=Q_{II}/2=Q_{III}}$$
$$= 1/(p_{11} + 2p_{12} + p_{13})$$
$$C_{IIA} = (Q_{II}/V_{II})_{Q_I=Q_{II}/2=Q_{III}} = 1/(p_{22} + p_{12})$$

$$
\begin{aligned}
C_{IB} &= C_{IIIB} = (Q_I/V_I)_{Q_I=-Q_{II}/2=Q_{III}} \\
&= 1/(p_{11} - 2p_{12} + p_{13}) \\
C_{IIB} &= (Q_{II}/V_{II})_{Q_I=-Q_{II}/2=Q_{III}} = 1/(p_{22} - p_{12}) \\
C_{IC} &= C_{IIIC} = (Q_I/V_I)_{Q_I=-Q_{III},Q_{II}=0} \\
&= 1/(p_{11} - p_{13})
\end{aligned}
\tag{2}
$$

where the subscripts I, II, and III denote the coupled conductors and the subscripts A, B, and C denote the fundamental modes.

When the coefficients of potential satisfy the following condition:

$$2p_{22} = p_{11} + p_{13}, \tag{3}$$

we obtain from (2)

$$
\begin{aligned}
C_{IIA} &= 2C_{IA} \\
C_{IIB} &= 2C_{IB}
\end{aligned}
\tag{4}
$$

and from (1)

$$
\begin{aligned}
&V_I = V_{II} = V_{III} && \text{for A-mode} \\
&V_I = -V_{II} = V_{III} && \text{for B-mode} \\
&V_I = -V_{III},\ V_{II} = 0 && \text{for C-mode}
\end{aligned}
\tag{5}
$$

where the V's are the potentials of the coupled conductors with respect to ground for the fundamental modes in Fig. 2.

Equation (3) is an important relation in this paper and a discussion of this condition is presented in Section VII.

B. Fundamental Modes for the Derivation of Admittance Matrix

Figure 3 shows a set of fundamental modes given in convenient forms for deriving the admittance matrix of the coupled transmission line six-port. They are again designated as A-, B-, and C-mode since it will be shown later that fundamental modes in Fig. 3 are identical to those in Fig. 2.

Solving (1) for the Q's, we obtain

$$
\begin{bmatrix} Q_I \\ Q_{II} \\ Q_{III} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{12} \\ c_{13} & c_{12} & c_{11} \end{bmatrix} \cdot \begin{bmatrix} V_I \\ V_{II} \\ V_{III} \end{bmatrix}
\tag{6}
$$

where c_{11} and c_{22} are the coefficients of self capacity and c_{12} and c_{13} are the coefficients of induction.

Using (6), the static capacitances of conductors to ground per unit length for the fundamental modes shown in Fig. 3 are given by

$$
\begin{aligned}
C_{IA} &= C_{IIIA} = c_{11} + c_{12} + c_{13} \\
C_{IIA} &= c_{22} + 2c_{12} \\
C_{IB} &= C_{IIIB} = c_{11} - c_{12} + c_{13} \\
C_{IIB} &= c_{22} - 2c_{12} \\
C_{IC} &= C_{IIIC} = c_{11} - c_{13}.
\end{aligned}
\tag{7}
$$

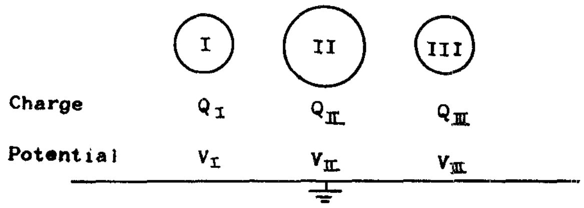


Fig. 1. Cross section of a coupled three-conductor transmission line above ground.

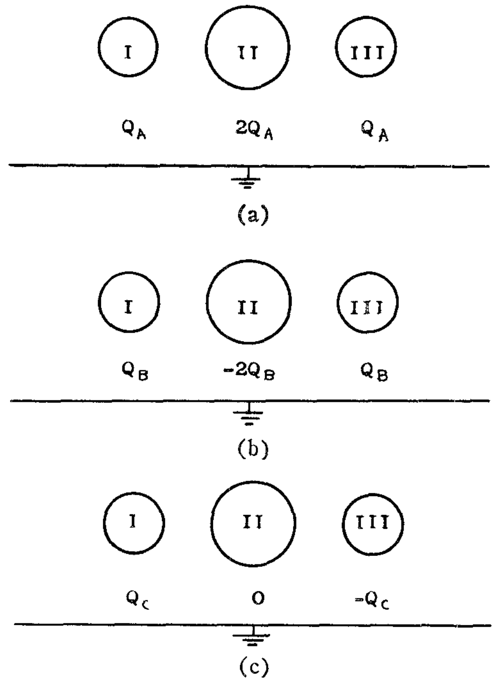


Fig. 2. Fundamental modes used in deriving the impedance matrix. (a) A-mode. (b) B-mode. (c) C-mode.

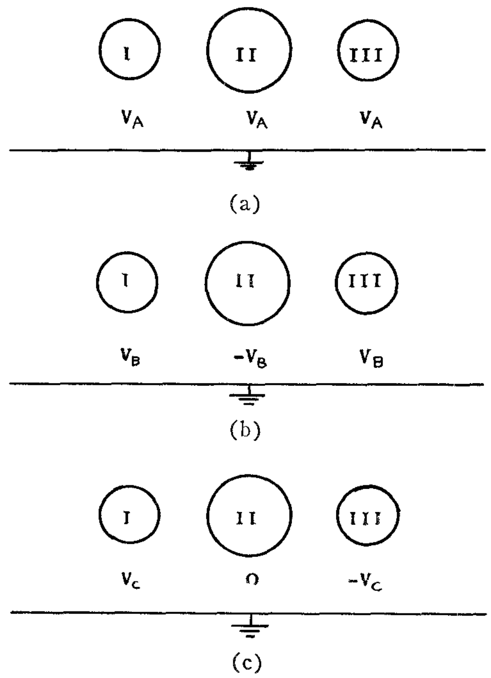


Fig. 3. Fundamental modes used in deriving the admittance matrix. (a) A-mode. (b) B-mode. (c) C-mode.

Equation (3) can be rewritten in terms of the c's in (6) as

$$c_{22} = 2(c_{11} + c_{13}). \tag{8}$$

Substituting (8) in (7) gives (4) as in the previous case. Furthermore, we obtain from (6) and (8)

$$Q_{\text{I}} = \frac{Q_{\text{II}}}{2} = Q_{\text{III}} \qquad \text{for A-mode}$$

$$Q_{\text{I}} = -\frac{Q_{\text{II}}}{2} = Q_{\text{III}} \qquad \text{for B-mode}$$

$$Q_{\text{I}} = -Q_{\text{III}},\ Q_{\text{II}} = 0 \qquad \text{for C-mode} \tag{9}$$

where the Q's are the charges on the conductors per unit length for the fundamental modes in Fig. 3.

Referring to (5) and (9), it can be seen that the fundamental modes in Fig. 2 are identical to those in Fig. 3 under the condition of (3). Therefore the characteristic impedances for the fundamental modes in Fig. 2 are simply the reciprocals of the corresponding characteristic admittances for those in Fig. 3.

III. Immittance Matrices of the Coupled Transmission Line Six-Port

A convenient way to describe the behavior of the coupled three-conductor transmission line above ground is by means of either impedance matrix or admittance matrix, both of which will be derived in this section.

A. Impedance Matrix

The configuration to be considered here is the parallel coupled transmission line six-port as shown in Fig. 4 where, for simplicity, the ground planes are removed. The electrical length of each coupled conductor is equal to θ. Characteristic impedances of each conductor to ground are designated as listed in Table I.

Since the fundamental modes chosen in the previous section are orthogonal, the impedance matrix for the configuration of Fig. 4 is derived by superposition. Corresponding to the fundamental modes in Fig. 2 and applying the sets of current generators at each port [12] as shown in Fig. 5, after some manipulations we obtain the impedance matrix,

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} P\alpha & R\alpha & S\alpha & S\beta & R\beta & P\beta \\ R\alpha & Q\alpha & R\alpha & R\beta & Q\beta & R\beta \\ S\alpha & R\alpha & P\alpha & P\beta & R\beta & S\beta \\ S\beta & R\beta & P\beta & P\alpha & R\alpha & S\alpha \\ R\beta & Q\beta & R\beta & R\alpha & Q\alpha & R\alpha \\ P\beta & R\beta & S\beta & S\alpha & R\alpha & P\alpha \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix} \tag{10}$$

where

$$P = (Z_{oa} + Z_{ob} + 2Z_{oc})/4$$
$$Q = (Z_{oA} + Z_{oB})/2$$
$$R = (Z_{oa} - Z_{ob})/4 = (Z_{oA} - Z_{oB})/2$$
$$S = (Z_{oa} + Z_{ob} - 2Z_{oc})/4, \tag{11}$$
$$\alpha = -j \cot \theta$$
$$\beta = -j \operatorname{cosec} \theta. \tag{12}$$

Characteristic immittances, Z_o and Y_o, of a lossless uniform transmission line operating in the TEM-mode can be related to its shunt capacitance C per unit length by [14]

$$\sqrt{\epsilon_r}\, Z_o = \frac{\sqrt{\epsilon_r}}{Y_o} = \frac{120\pi}{(C/\epsilon)} \tag{13}$$

where ϵ and ϵ_r are the permittivity and the relative permittivity of the dielectric medium, respectively. From (4) and (13), referring to Table I, we obtain

$$2Z_{oA} = Z_{oa}$$
$$2Z_{oB} = Z_{ob}. \tag{14}$$

It should be noted that, among five characteristic impedances in Table I, three are independent. Then (11) reduces to

$$P = (Z_{oA} + Z_{oB} + Z_{oc})/2$$
$$Q = (Z_{oA} + Z_{oB})/2$$
$$R = (Z_{oA} - Z_{oB})/2$$
$$S = (Z_{oA} + Z_{oB} - Z_{oc})/2, \tag{15}$$

and

$$2Q = P + S. \tag{16}$$

Equation (16) corresponds to (3).

B. Admittance Matrix

The admittance matrix of the coupled transmission line six-port shown in Fig. 6 is derived in an analogous manner. Characteristic admittances of each conductor to ground are designated as listed in Table II. In this case, the fundamental modes are excited by the sets of voltage generators [12] as shown in Fig. 7. Then the admittance matrix for the configuration of Fig. 6 is

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix} = \begin{bmatrix} P'\alpha & R'\alpha & S'\alpha & S'\beta & R'\beta & P'\beta \\ R'\alpha & Q'\alpha & R'\alpha & R'\beta & Q'\beta & R'\beta \\ S'\alpha & R'\alpha & P'\alpha & P'\beta & R'\beta & S'\beta \\ S'\beta & R'\beta & P'\beta & P'\alpha & R'\alpha & S'\alpha \\ R'\beta & Q'\beta & R'\beta & R'\alpha & Q'\alpha & R'\alpha \\ P'\beta & R'\beta & S'\beta & S'\alpha & R'\alpha & P'\alpha \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} \tag{17}$$

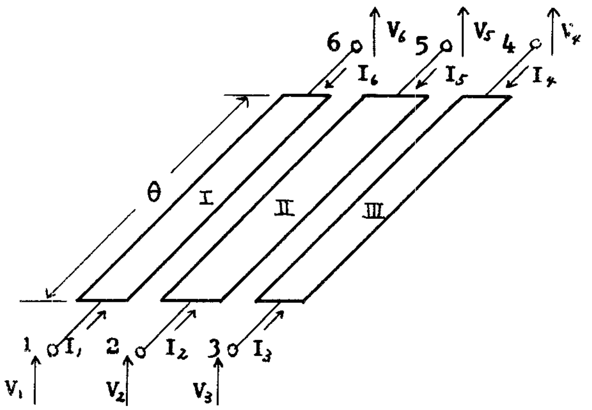


Fig. 4. Voltages and currents for the impedance matrix.

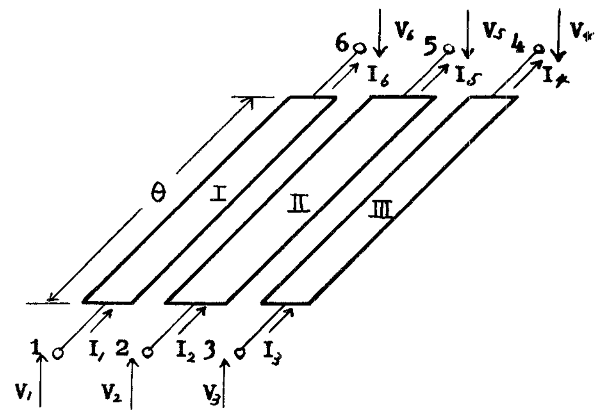


Fig. 6. Voltages and currents for the admittance matrix.

TABLE I
CHARACTERISTIC IMPEDANCES

Fundamental Mode	Conductor		
	I	II	III
A-mode	Z_{oa}	Z_{oA}	Z_{oa}
B-mode	Z_{ob}	Z_{oB}	Z_{ob}
C-mode	Z_{oc}	—	Z_{oc}

TABLE II
CHARACTERISTIC ADMITTANCES

Fundamental Mode	Conductor		
	I	II	III
A-mode	Y_{oa}	Y_{oA}	Y_{oa}
B-mode	Y_{ob}	Y_{oB}	Y_{ob}
C-mode	Y_{oc}	—	Y_{oc}

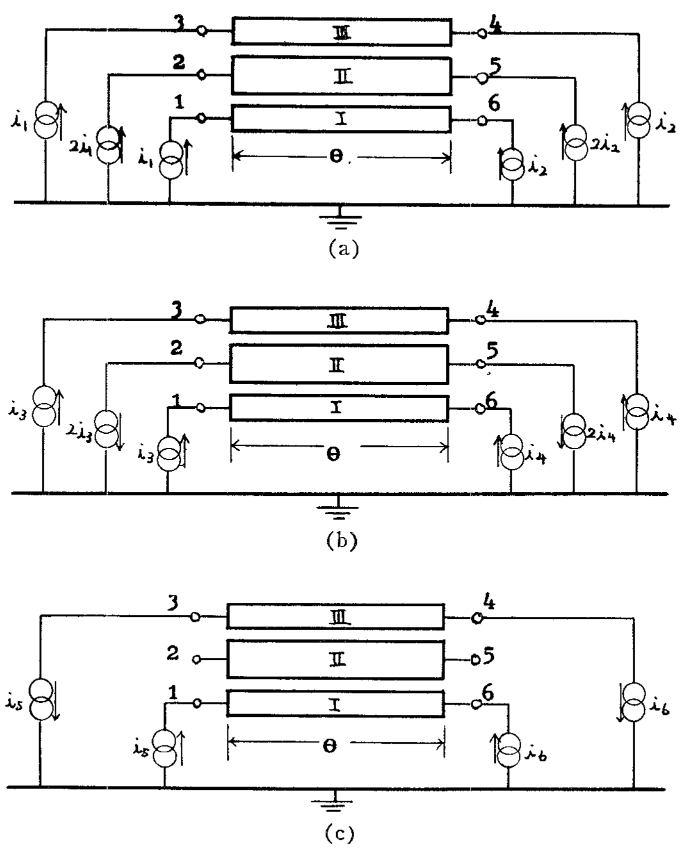


Fig. 5. Excitations of fundamental modes by the current generators. (a) A-mode. (b) B-mode. (c) C-mode.

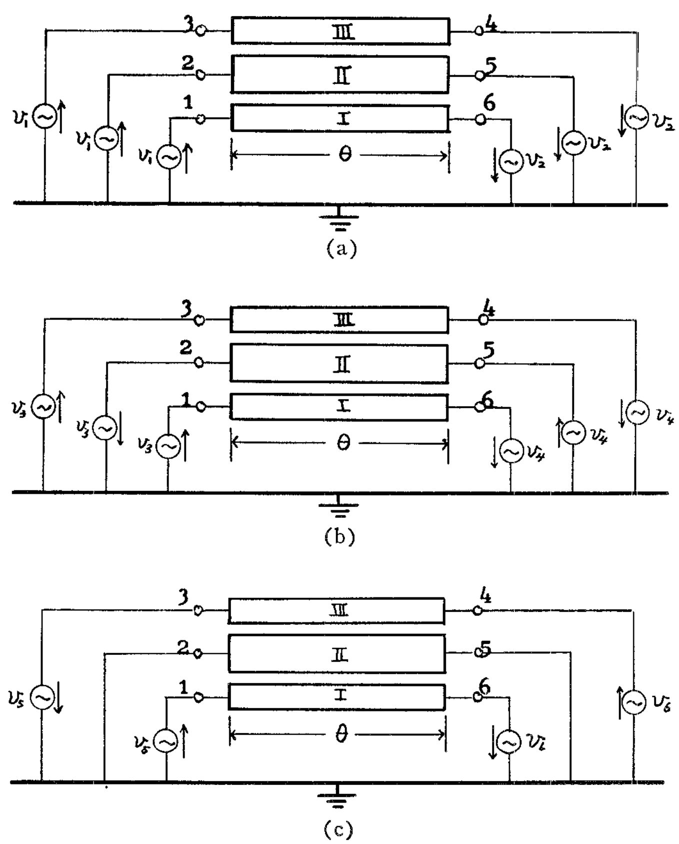


Fig. 7. Excitations of fundamental modes by the voltage generators. (a) A-mode. (b) B-mode. (c) C-mode.

where

$$P' = (Y_{oa} + Y_{ob} + 2Y_{oc})/4$$
$$Q' = (Y_{oA} + Y_{oB})/2$$
$$R' = (Y_{oa} - Y_{ob})/2 = (Y_{oA} - Y_{oB})/4$$
$$S' = (Y_{oa} + Y_{ob} - 2Y_{oc})/4,$$

α and β are given by (12). (18)

From (4) and (13), referring to Table II, we obtain

$$Y_{oA} = 2Y_{oa}$$
$$Y_{oB} = 2Y_{ob}. \quad (19)$$

Then (18) reduces to

$$P' = (Y_{oa} + Y_{ob} + 2Y_{oc})/4$$
$$Q' = (Y_{oa} + Y_{ob})$$
$$R' = (Y_{oa} - Y_{ob})/2$$
$$S' = (Y_{oa} + Y_{ob} - 2Y_{oc})/4, \quad (20)$$

and

$$Q' = 2(P' + S'). \quad (21)$$

Equation (21) corresponds to (18).

IV. Design Formulas for the Coupled Strip Transmission Line with Three Center Conductors

Typical cross sections of coupled strip transmission lines with three center conductors are shown in Fig. 8 where the thin center conductors are shielded by the upper and lower ground planes of infinite width.

These three configurations use thin strips parallel to the ground planes, and thus are applicable to the printed-circuit constructions. The coplanar configuration of Fig. 8(a) was previously analyzed and the design formula is available [13], while the cross sections of Figs. 8(b) and 8(c) are treated in this paper.

As was pointed out [13], an additional degree of freedom is obtained by using the cross sections of Figs. 8(b) and 8(c) as required in many practical cases.

In Figs. 8(b) and 8(c), strip II is divided into two parts denoted by II-1 and II-2 which are parallel to the ground planes. Let strips II-1 and II-2 always be at the same potential; they will then form a single transmission line coupled to the transmission line formed by strip I and to that formed by strip III [14]. Of course, coupling between strips I and III cannot be neglected.

In the design procedure, characteristic immittances are first tabulated to provide the desired circuit performance. Then the corresponding physical dimensions are determined. Design formulas for these two configurations containing zero-thickness conductors are derived rigorously by means of conformal mapping techniques. Only the results obtained are presented in Table III. The details of the analysis are given in the Appendix. Of course, the condition of (3), i.e., (14) or (19), is involved in the design formulas in Table III [see (51) and (52) in the Appendix], and raises no limitation on coupling.

By the use of the design formulas, cross sections are designed in a straightforward process to have the desired characteristic immittances. Calculations may be carried out with the aid of the available tables or rapidly converging ϑ functions.

Moderate coupling between strips I and III may be achieved if the cross section of Fig. 8(b) is used, while the cross section of Fig. 8(c) is suitable when closer coupling between strips I and III is required.

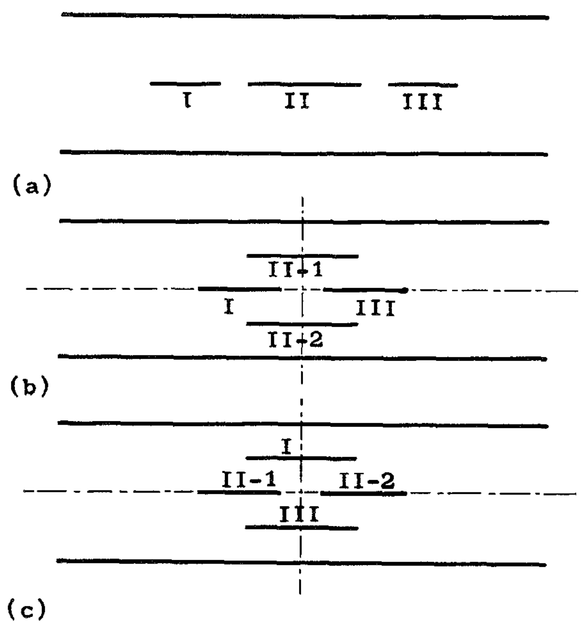
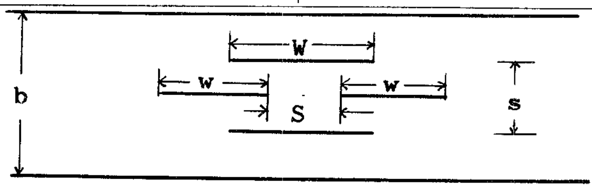


Fig. 8. Typical cross sections of coupled strip transmission line with three center conductors.

TABLE III

DESIGN FORMULAS FOR THE CROSS SECTIONS OF FIGS. 8(b) AND 8(c)

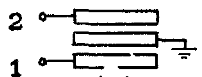
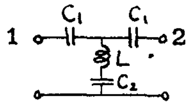
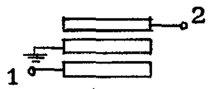
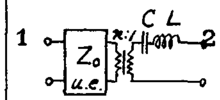
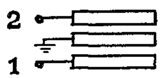
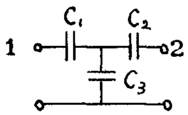
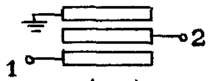
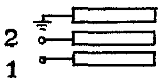
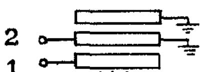
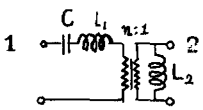
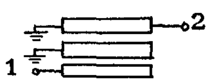
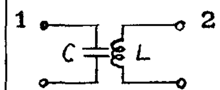
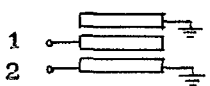
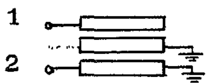
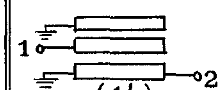
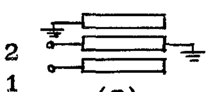
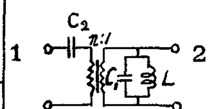
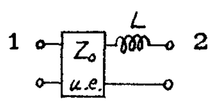
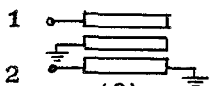
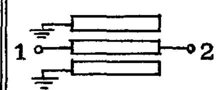
Cross Section	Fig. 8(b)	Fig. 8(c)
Dimensions		
	Characteristic Immittances	
A-mode	$Z_{oa} = 2Z_{oA} = 2/Y_{oA} = 1/Y_{oa} = \dfrac{30\pi}{\sqrt{\epsilon_r}} \dfrac{K'(k_a)}{K(k_a)}$	
B-mode	$Z_{ob} = 2Z_{oB} = 2/Y_{oB} = 1/Y_{ob} = \dfrac{30\pi}{\sqrt{\epsilon_r}} \dfrac{K'(k_b)}{K(k_b)}$	
C-mode	$Z_{oc} = 1/Y_{oc} = \dfrac{30\pi}{\sqrt{\epsilon_r}} \dfrac{K'(k_c)}{K(k_c)}$	$Z_{oc} = 1/Y_{oc} = \dfrac{60\pi}{\sqrt{\epsilon_r}} \dfrac{K'(k_c)}{K(k_c)}$
Substitution	$\lambda = \sqrt{k_a k_b}, \quad \mu = \sqrt{\dfrac{k_a}{k_b}}, \quad \nu = k_c$	
	$\lambda^* = \dfrac{1-\lambda}{1+\lambda}, \quad \mu^* = \dfrac{1-\mu}{1+\mu}, \quad \nu^* = \dfrac{1-\nu}{1+\nu}$	
modulus	$k = \sqrt{\dfrac{(1-\lambda^{*2})(\lambda^{*2}-\nu^{*2})}{\lambda^{*2}(1-\mu^{*2})-\nu^{*2}(1-\lambda^{*2}\mu^{*2})}}$	$k = \nu$
Parameters	$a = \mathrm{sn}^{-1}\left[\dfrac{1}{k}\sqrt{1-\lambda^{*2}}\right]$ $\xi = \mathrm{sn}^{-1}\left[\dfrac{1}{k}\sqrt{\dfrac{1-\lambda^{*2}}{1-\mu^{*2}}}\right]$ $\eta = \mathrm{sn}^{-1}\left[\dfrac{1}{k}\sqrt{\dfrac{1-\lambda^{*2}}{1-\lambda^{*2}\mu^{*2}}}\right]$	$a = \mathrm{sn}^{-1}\left[\dfrac{1}{k}\dfrac{2\sqrt{\lambda}}{1+\lambda}\right]$ $\xi = \mathrm{sn}^{-1}\left[\dfrac{1}{k}\dfrac{1+u}{1+\lambda}\sqrt{\dfrac{\lambda}{\mu}}\right]$ $\eta = \mathrm{sn}^{-1}\left[\dfrac{1}{k}\dfrac{(1+\mu)\sqrt{\lambda}}{\sqrt{(1+\lambda\mu)(\lambda+\mu)}}\right]$
	$\zeta = \mathrm{sn}^{-1}\left[\dfrac{1}{k}\dfrac{\sqrt{Z(a)}}{\sqrt{Z(a)\mathrm{sn}^2 a + \mathrm{sn}\, a\, \mathrm{cn}\, a\, \mathrm{dn}\, a}}\right]$	
Dimensions	$\dfrac{W}{b} = \dfrac{1}{\pi}\log\dfrac{\Theta(\zeta+a)}{\Theta(\zeta-a)},$ $\dfrac{w}{b} = \dfrac{1}{2\pi}\log\dfrac{H(\xi-a)H(\eta+a)}{H(\xi+a)H(\eta-a)},$	$\dfrac{s}{b} = \dfrac{a}{K(k)}$ $\dfrac{S}{b} = \dfrac{1}{\pi}\log\dfrac{H(\xi+a)}{H(\xi-a)}$

Notes: $\Theta(u)$ = Jacobian theta-function.
$H(u)$ = Jacobian eta-function.
$Z(u)$ = Jacobian zeta-function.
sn u, cn u, and dn u = Jacobian elliptic functions.
$K(k)$ and $K'(k)$ = Complete elliptic integrals of the first kind.

$$\mathrm{sn}^{-1}x = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = F(x, k) = \text{Elliptic integral of the first kind.}$$

ϵ_r = Relative permittivity of the dielectric medium filling the cross section.

TABLE IV
EQUIVALENT CIRCUITS

Original Circuit	Equivalent Circuit	Element Value	Original	Equivalent Circuit	Element Value
 (1)		$C_1=\frac{1}{P-S}$, $C_2=\frac{Q}{QS-R^2}$ $L=\frac{R^2}{Q}$	 (9)		$Zo=\frac{PQ-R^2}{Q}$, $n=\frac{PQ-R^2}{QS-R^2}$ $C=\frac{PQ-R^2}{2(P-S)(Q^2-R^2)}$, $L=\frac{R^2}{Q}$
 (2)		$C_1=C_2=\frac{1}{P-S}$ $C_3=\frac{Q}{QS-R^2}$	 (10)		$Zo=\frac{2Q(P-S)}{P}$, $n=\frac{2Q}{R}$ $C=\frac{Q}{Q^2-R^2}$, $L=\frac{R^2}{P}$
 (3)		$C_1=\frac{P}{(P-S)(P+S-R)}$, $C_3=\frac{P}{R(P-S)}$ $C_2=\frac{P}{(PQ-R^2)-R(P-S)}$	 (11)		$Zo=\frac{PQ-R^2}{P}$, $n=\frac{PQ-R^2}{R(P-S)}$ $C=\frac{PQ-R^2}{2(P-S)(Q^2-R^2)}$, $L=\frac{S^2}{P}$
 (4)		$C=\frac{PQ-R^2}{2(P-S)(Q^2-R^2)}$, $L_2=Q$ $L_1=\frac{(QS-R^2)^2}{Q(PQ-R^2)}$, $n=\frac{R}{Q}$	 (12)		$C=\frac{PQ-R^2}{2(P-S)(Q^2-R^2)}$ $L=P$
 (5)		$C=\frac{Q}{Q^2-R^2}$, $L_2=P$ $L_1=\frac{(P-Q)R^2}{PQ}$, $n=\frac{R}{P}$	 (13)		$C=\frac{PQ-R^2}{2(P-S)(Q^2-R^2)}$ $L=Q$
 (6)		$C=\frac{PQ-R^2}{2(P-S)(Q^2-R^2)}$, $L_2=P$ $L_1=\frac{R^2(P-S)^2}{P(PQ-R^2)}$, $n=\frac{S}{P}$	 (14)		$C=\frac{Q}{Q^2-R^2}$ $L=P$
 (7)		$C_1=\frac{R^2}{Q(PQ-R^2)}$, $L=Q$ $C_2=\frac{PQ-R^2}{2(P-S)(Q^2-R^2)}$, $n=\frac{R(P-S)}{PQ-R^2}$	 (15)		$Zo=\frac{2(P-S)(Q^2-R^2)}{PQ-R^2}$ $L=\frac{PR^2+QS^2-2SR^2}{PQ-R^2}$
 (8)		$C_1=\frac{R^2}{P(PQ-R^2)}$, $L=P$ $C_2=\frac{PQ-R^2}{2(P-S)(Q^2-R^2)}$, $n=\frac{QS-R^2}{PQ-R^2}$	 (16)		$Zo=\frac{Q^2-R^2}{Q}$ $L=\frac{R^2}{Q}$

V. Equivalent Circuits

The immittance matrices derived in Section III may serve as a basis for analyzing the transmission properties of coupled strip transmission lines with three center conductors (or other coupled TEM three-conductor transmission lines with common return). In many practical cases, however, it is desirable to present the equivalent circuits of the two-port networks obtained by applying the pertinent port conditions to the coupled transmission line six-port [15], [16].

Distributed TEM networks composed of lumped resistors and lossless equal length transmission lines can be treated in a manner analogous to lumped constant networks by means of a frequency transformation [9]

$$s = j \cdot \tan (\pi f/2f_0) \tag{22}$$

where f_0 is the real constant frequency at which a transmission line is a quarter-wavelength long and f is the real frequency variable.

The canonical section to be considered here is the coupled strip transmission line with three center conductors in which each component line is a quarter-wavelength long at f. Then substituting

$$s = j \cdot \tan (\pi f/2f_0) = j \cdot \tan \theta$$

in (10) gives the impedance matrix of the canonical section

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \frac{1}{s} \begin{bmatrix} P & R & S & S\sqrt{1-s^2} & R\sqrt{1-s^2} & P\sqrt{1-s^2} \\ R & Q & R & R\sqrt{1-s^2} & Q\sqrt{1-s^2} & R\sqrt{1-s^2} \\ S & R & P & P\sqrt{1-s^2} & R\sqrt{1-s^2} & S\sqrt{1-s^2} \\ S\sqrt{1-s^2} & R\sqrt{1-s^2} & P\sqrt{1-s^2} & P & R & S \\ R\sqrt{1-s^2} & Q\sqrt{1-s^2} & R\sqrt{1-s^2} & R & Q & R \\ P\sqrt{1-s^2} & R\sqrt{1-s^2} & S\sqrt{1-s^2} & S & R & P \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix} \tag{23}$$

where P, Q, R, and S are given by (15) in Section III. The corresponding admittance matrix is easily determined by replacing in (23) P by P', Q by Q', R by R', S by S', $[V]$ by $[I]$ and $[I]$ by $[V]$, where P', Q', R', and S' are given by (20).

Various s-plane equivalent circuits are obtained from the immittance matrices of the canonical section for various port conditions.

The two-port networks, in which one or two component lines are either open-circuited or short-circuited at both ends, are not considered in this section, since these component lines do not contribute to the transmission properties but only change the immittance levels of the networks. These networks, however, may be applicable to the asymmetrical coupled strip transmission line with two center conductors as will be discussed in the next section.

A list of sixteen kinds of circuit configurations, their corresponding equivalent circuits and the relationships of element values is given in Table IV where the boxes represent the unit elements. Only the two-port networks with up to two short-circuited ports, whose equivalent circuits have been obtained from the impedance matrix, are presented in this table. When the number of the short-circuited ports is equal to three, the use of the admittance matrix will simplify the derivation of the equivalent circuits and the different circuits may be obtained.

As a specific example of the main advantages gained by the exact analysis, consider the third order interdigital network with open-circuited terminating lines, whose equivalent circuit is the m-derived type of high-pass filter, as shown in Table IV (1). However, if the coupling between nonadjacent lines is neglected, the capacitor in the shunt series-resonant arm vanishes, that is, its approximate equivalent circuit is expressed as the usual constant-k type of high-pass filter [8]. Thus we find that it is useful for making the cutoff characteristics steeper to take into account the coupling between nonadjacent lines.

Equivalent circuits in Table IV can be used to design the strip-line networks (or other TEM networks) from the lumped constant networks by the similar method described by Ozaki and Ishii [9], and Wenzel [10] for the case of coupled strip transmission lines with two center conductors. Of course, these equivalent circuits may be applicable to synthesis for both broad and narrow bandwidth, since the circuit equivalences and identities are theoretically valid over the entire frequency spectrum.

VI. Application to the Asymmetrical Coupled Strip Transmission Line with Two Center Conductors

Let us now consider the canonical section of the coupled strip transmission line with three center conductors. When the line III is either open-circuited or short-circuited at both ends, the resulting four-port network is considered to be equivalent to the canonical section of the asymmetrical coupled strip transmission line with two center conductors.

While several kinds of existing strip-line filters employ the asymmetrical coupled strip transmission line composed of two center conductors of equal length as the canonical section [9], [10], exact design formulas for the cross section have not been given even for the thin strip case. Approximate formulas [17], [18] are available for the practical cases, but they cannot be used when the ratio of the two strip-widths becomes relatively greater (or smaller) than unity, or when the extremely high characteristic impedances (or low characteristic admittances) are required.

In this section, it will be shown that exact design formulas for the asymmetrical coupled strip transmission line with two center conductors can be obtained from those for the coupled strip transmission line with three center conductors presented in Section IV.

A. Open-Circuited Case

The impedance matrix of the four-port network, in which the line III is open-circuited at both ends is, using the notation in Fig. 9(a),

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \frac{1}{s}\begin{bmatrix} P & R & R\sqrt{1-s^2} & P\sqrt{1-s^2} \\ R & Q & Q\sqrt{1-s^2} & R\sqrt{1-s^2} \\ R\sqrt{1-s^2} & Q\sqrt{1-s^2} & Q & R \\ P\sqrt{1-s^2} & R\sqrt{1-s^2} & R & P \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} \tag{24}$$

where P, Q, and R are given by (15), and s by (22).

On the other hand, the impedance matrix of the canonical section of the asymmetrical coupled strip transmission line with two center conductors shown in Fig. 9(b) is [9]

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \frac{1}{s}\begin{bmatrix} \frac{Z_{oe}^a+Z_{oo}^a}{2} & \frac{Z_{oe}^a-Z_{oo}^a}{2} & \frac{Z_{oe}^a-Z_{oo}^a}{2}\sqrt{1-s^2} & \frac{Z_{oe}^a+Z_{oo}^a}{2}\sqrt{1-s^2} \\ \frac{Z_{oe}^b-Z_{oo}^b}{2} & \frac{Z_{oe}^b+Z_{oo}^b}{2} & \frac{Z_{oe}^b+Z_{oo}^b}{2}\sqrt{1-s^2} & \frac{Z_{oe}^b-Z_{oo}^b}{2}\sqrt{1-s^2} \\ \frac{Z_{oe}^b-Z_{oo}^b}{2}\sqrt{1-s^2} & \frac{Z_{oe}^b+Z_{oo}^b}{2}\sqrt{1-s^2} & \frac{Z_{oe}^b+Z_{oo}^b}{2} & \frac{Z_{oe}^b-Z_{oo}^b}{2} \\ \frac{Z_{oe}^a+Z_{oo}^a}{2}\sqrt{1-s^2} & \frac{Z_{oe}^a-Z_{oo}^a}{2}\sqrt{1-s^2} & \frac{Z_{oe}^a-Z_{oo}^a}{2} & \frac{Z_{oe}^a+Z_{oo}^a}{2} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} \tag{25}$$

where

Z_{oe}^a (Z_{oe}^b) is the characteristic impedance of strip $a(b)$ to ground with equal currents in the same direction (even-mode characteristic impedance),

Z_{oo}^a (Z_{oo}^b) is that with equal currents in the opposite direction (odd-mode characteristic impedance),

and

$$Z_{oe}^a - Z_{oo}^a = Z_{oe}^b - Z_{oo}^b.$$

Comparing (25) with (24) shows these equations to be equivalent, if

$$\begin{aligned} P &= (Z_{oe}^a + Z_{oo}^a)/2 \\ Q &= (Z_{oe}^b + Z_{oo}^b)/2 \\ R &= (Z_{oe}^a - Z_{oo}^a)/2 = (Z_{oe}^b - Z_{oo}^b)/2. \end{aligned} \tag{26}$$

From (15) and (26) we obtain

$$\begin{aligned} Z_{oA} &= Z_{oe}^b \\ Z_{oB} &= Z_{oo}^b \\ Z_{oc} &= 2(Z_{oe}^a - Z_{oe}^b). \end{aligned} \tag{27}$$

Z_{oA}, Z_{oB}, and Z_{oc} can be evaluated from the desired values of Z_{oe}^a, Z_{oo}^a, Z_{oe}^b, and Z_{oo}^b. Then the cross section is designed by means of the exact design formulas in Table III.

In this case, equivalent circuits of the two-port networks for various port conditions are listed in Table V. While these equivalent circuits are the same as those presented in [9] and [10], the element values are expressed as functions of the characteristic impedances of the coupled strip transmission line treated in this paper in order to utilize the design formulas in Table III.

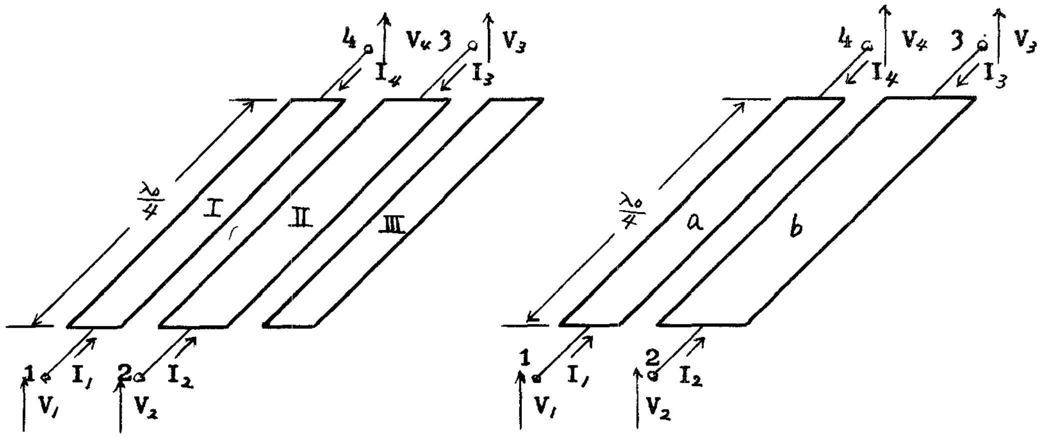
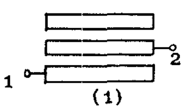
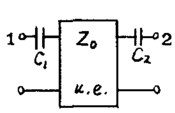
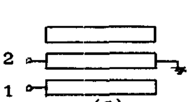
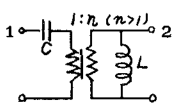
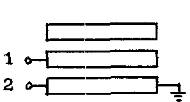
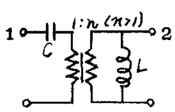
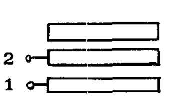
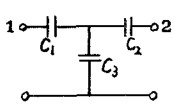
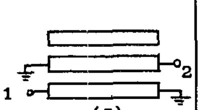
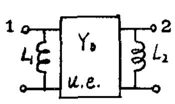
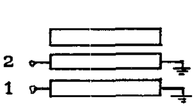
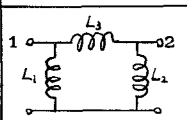


Fig. 9. (a) Canonical section of coupled strip transmission line with three center conductors in which the line III is open-circuited at both ends. (b) Canonical section of asymmetrical coupled strip transmission line with two center conductors.

TABLE V

EQUIVALENT CIRCUITS (OPEN-CIRCUITED CASE)

Original Circuit	Equivalent Circuit	Element Value
 (1)		P = Zo + 1/C₁ Q = Zo + 1/C₂ R = Zo
 (2)		P = 1/C + L/n² Q = L R = L/n n² > LC(n² -1)
 (3)		P = L Q = 1/C + L/n² R = L/n LC(n² -1) > n² > LC(n-1)
 (4)		P = 1/C₁ + 1/C₃ Q = 1/C₂ + 1/C₃ R = 1/C₃
 (5)		Q/T = Yo + 1/L₁ P/T = Yo + 1/L₂ R/T = Yo
 (6)		Q/T = 1/L₁ + 1/L₃ P/T = 1/L₂ + 1/L₃ R/T = 1/L₃

Notes: P, Q, and R are given by (15).
$T = PQ - R^2$

B. Short-Circuited Case

Figure 10(a) shows the canonical section of the coupled strip transmission line with three center conductors, in which the line III is short-circuited at both ends, while Fig. 10(b) shows that of the asymmetrical coupled strip transmission line with two center conductors.

In this case, the use of the admittance matrix instead of the impedance matrix simplifies the procedure.

Comparing the four-port admittance matrix of Fig. 10(a) with that of Fig. 10(b) [9] shows that these two networks are equivalent, if

$$P' = (Y_{oe}^a + Y_{oo}^a)/2$$
$$Q' = (Y_{oe}^b + Y_{oo}^b)/2$$
$$R' = (Y_{oe}^a - Y_{oo}^a)/2 = (Y_{oe}^b - Y_{oo}^b)/2 \qquad (28)$$

where

P', Q', and R' are given by (20),
Y_{oe}^a (Y_{oe}^b) is the even-mode characteristic admittance of strip $a(b)$ to ground,
Y_{oo}^a (Y_{oo}^b) is the odd-mode characteristic admittance of strip $a(b)$ to ground.

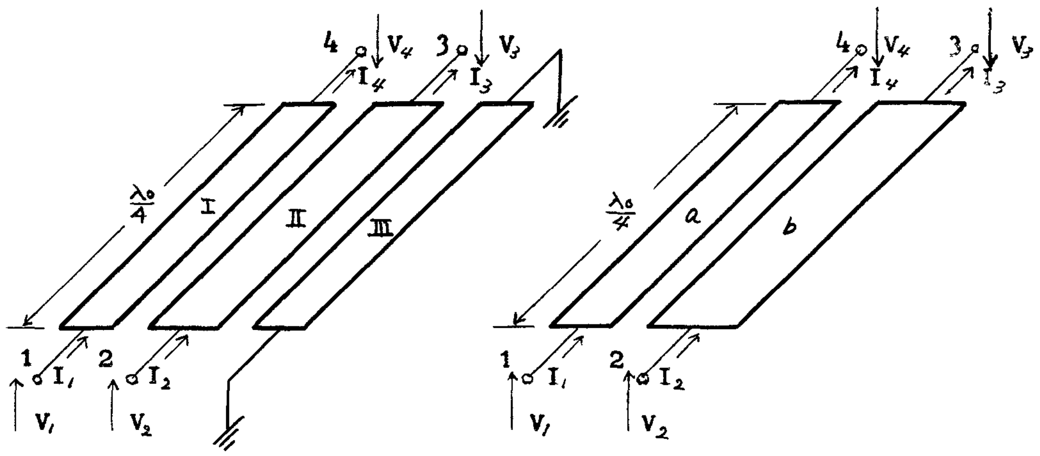
From (20) and (28) we obtain

$$Y_{oa} = (3Y_{oe}^b - Y_{oo}^b)/4$$
$$Y_{ob} = (3Y_{oo}^b - Y_{oe}^b)/4$$
$$Y_{oc} = (Y_{oe}^a + Y_{oo}^a) - (Y_{oe}^b + Y_{oo}^b)/4. \qquad (29)$$

Equivalent circuits and element values for this case are listed in Table VI.

VII. DISCUSSION

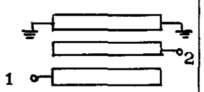
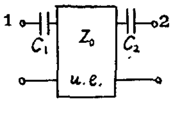
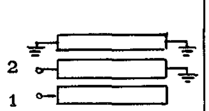
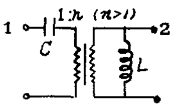
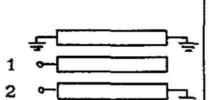
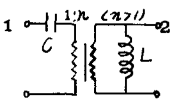
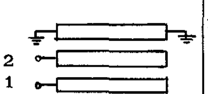
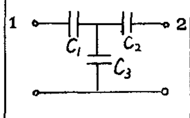
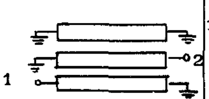
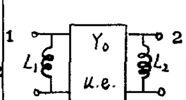
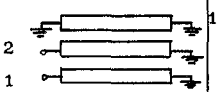
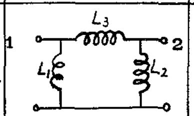
As stated in Section III, among the five characteristic impedances or admittances listed in Table I or II,



(a) (b)

Fig. 10. (a) Canonical section of coupled strip transmission line with three center conductors in which the line III is short-circuited at both ends. (b) Canonical section of asymmetrical coupled strip transmission line with two center conductors.

TABLE VI
EQUIVALENT CIRCUITS (SHORT-CIRCUITED CASE)

Original Circuit	Equivalent Circuit	Element Value
 (1)		$Q'/T'=Z_o+1/C_1$ $P'/T'=Z_o+1/C_2$ $-R'/T'=Z_o$
 (2)		$Q'/T'=1/C + L/n^2$ $P'/T'= L$ $-R'/T'= L/n$ $n^2>LC(n^2-1)$
 (3)		$Q'/T'= L$ $P'/T'=1/C + L/n^2$ $-R'/T'= L/n$ $LC(n^2-1)>n^2>LC(n-1)$
 (4)		$Q'/T'= 1/C_1+ 1/C_3$ $Q'/T'= 1/C_2+ 1/C_3$ $-R'/T'= 1/C_3$
 (5)		$P'= Y_o + 1/L_1$ $Q'= Y_o + 1/L_2$ $-R'= Y_o$
 (6)		$P'= 1/L_1+ 1/L_3$ $Q'= 1/L_2+ 1/L_3$ $-R'= 1/L_3$

Notes: P', Q', and R' are given by (20).

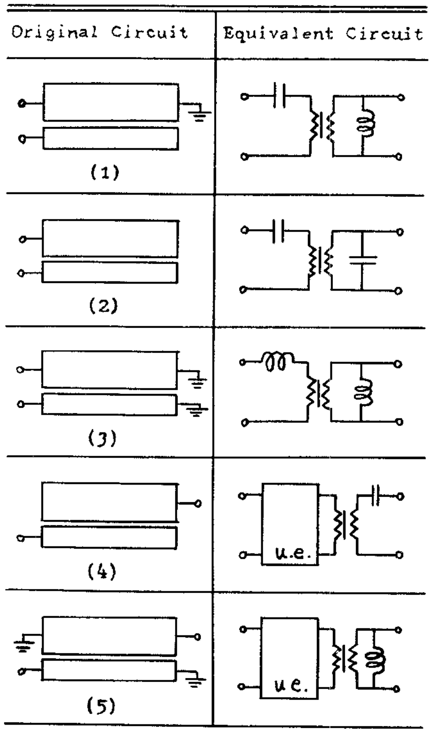
$T' = P'Q' - R'^2$

three are independent because of the existence of the condition of (3), for which we consider in this section.

A. On the Design of Distributed Networks by the Use of the Lumped Element Techniques

Equivalent circuits in Table IV contain at most four elements such as inductor, capacitor, etc., and there exists an ideal transformer in each of those containing four elements, as shown in Table IV (4) through (11). Thus, for such circuits, if we select the inductor, capacitor and unit-element to have the values required from the synthesis procedure, then the value of the turns ratio of the ideal transformer is exclusively fixed, since the coupled transmission lines treated in this paper have three independent characteristic impedances or admittances. However, this is not so restrictive if the network to be designed is symmetrical about its center. The reason is that for the symmetrical networks, the ideal transformers, which must be added for making use of the circuit equivalences in Table IV, always have the same turns ratio, but of opposite sense, and cancel each other. On the other hand, if the network to be designed is not symmetrical, the added transformers generally do not cancel and, therefore, one more ideal transformer must be added to preserve the property of the given network. This additional ideal transformer may be realized by constructing it to be contained in the circuits having an ideal transformer in their equivalent circuits, such as those in Table VII. The equivalent circuit representations in Table VII (2) through (5) are different from those in [9] or Tables V and VI in this paper. However Kuroda's identities and some network transformations guarantee these circuit equivalences. Circuit configurations in Table VII make use of the asymmetrical coupled strip transmission line with two center conductors having three independent char-

TABLE VII
EQUIVALENT CIRCUITS CONTAINING AN IDEAL TRANSFORMER



acteristic impedances or admittances, for which exact analysis has been made in this paper. Thus, whether the network to be designed is symmetrical or not, it is found that it may be designed by using the results presented in this paper. Although the turns ratio of the ideal transformers cannot be chosen arbitrarily, several physical configurations having identical responses often allow the desired networks to be realized within the practical range of the characteristic immittances, by the proper choice. The advantage gained by setting the condition of (3) is that the rigorous results can be obtained throughout the analysis, and, therefore, no limitation exists on immittance levels or on coupling.

B. On the Coupled Three-Conductor Transmission Line Above Ground Without the Condition of (3)

If we desire four independent characteristic impedances or admittances for the coupled three-conductor transmission line with common return, analysis must be done without the condition of (3).

Let us now consider the coupled three-conductor line above ground, symmetrical about the vertical centerline as in Fig. 1, but without the condition of (3). The mode conditions of the three orthogonal TEM-modes which can propagate on such a structure are as follows:

$$\text{A-mode} \qquad V_{\text{I}} = V_{\text{II}} = V_{\text{III}} \tag{30}$$

$$\text{B-mode:} \qquad Q_{\text{I}} = -Q_{\text{II}}/2 = Q_{\text{III}} \tag{31}$$

$$\text{C-mode:} \quad \begin{cases} Q_{\text{I}} = -Q_{\text{III}}, & Q_{\text{II}} = 0 \quad (32) \\ V_{\text{I}} = -V_{\text{III}}, & V_{\text{II}} = 0, \quad (33) \end{cases}$$

where the Q's are the charges on the conductors per unit length, the V's are the potentials of the conductors, subscripts I, II and III denote the coupled conductors, and (32) and (33) are equivalent.

Next, let the characteristic immittances for these three modes be designated as in Tables I and II in Section III. The characteristic impedances are, of course, the reciprocals of the corresponding characteristic admittances. Referring to the mode conditions of (30) through (33) and using (1), (6), and (13) yield

$$\frac{Z_{oa}}{Z_{oA}} = \frac{Z_{ob}}{Z_{oB}} = \frac{Y_{oA}}{Y_{oa}} = \frac{Y_{oB}}{Y_{ob}}. \tag{34}$$

Thus, it can be seen that among five characteristic impedances or admittances, four are independent. If we define the characteristic immittance ratio, γ, as

$$\gamma = \frac{Z_{oa}}{2Z_{oA}} = \frac{Z_{ob}}{2Z_{oB}} = \frac{Y_{oA}}{2Y_{oa}} = \frac{Y_{oB}}{2Y_{ob}}, \tag{35}$$

then the remaining relationships for these three modes are found to be

$$Q_{\text{I}} = Q_{\text{II}}/2\gamma = Q_{\text{III}} \qquad \text{for A-mode} \tag{36}$$

$$V_{\text{I}} = -\gamma V_{\text{II}} = V_{\text{III}} \qquad \text{for B-mode.} \tag{37}$$

Equations (30)–(33), (36), and (37) represent the complete mode conditions of the fundamental modes for the symmetrical coupled three-conductor transmission line above ground without the condition of (3). It should be noted that setting $\gamma = 1$ corresponds to the condition of (3), and we find that the coupled transmission line treated in this paper is considered the special case of the coupled three conductor transmission line without the condition of (3). Immittance matrices may be derived in a manner analogous to that used in this paper. Orthogonal mode representations of (31), (32), and (36) are convenient for the derivation of the impedance matrix, while those of (30), (33), and (37) are suitable for the admittance matrix.

Furthermore, if we employ the line configurations in Figs. 8(b) and 8(c) for this case, design formulas may be derived by the similar method described in the Appendix. The procedure and mapping functions are exactly the same for this case as for the coupled transmission line treated in this paper, except that, for A and B-mode, the final mapping plane, i.e., χ-plane in Fig. 12(c), becomes the upper half of the coupled strip transmission line with two unequal-width strips, for which exact analysis has not yet been made, instead of two equal-width strips. Although the exact design formulas cannot be obtained for this case, they can be derived approximately by using the Ishii's results [17] giving the approximate formulas for the coupled strip transmission line with two unequal-width strips. In such a manner analysis may be made for the coupled three-conductor transmission line without the condition of (3). Work in this area is continuing.

VIII. Conclusions

Basic information has been presented for a new type of coupled strip transmission line which, together with a more conventional one containing two center conductors, may have a wide variety of applications in the design of various microwave components. Six-port immittance matrices and design formulas for the cross sections based on the rigorous conformal mapping solutions may serve as bases for analysis and design of the devices using coupled strip transmission lines with three center conductors. The line configurations proposed are well suited to the printed-circuit constructions. Equivalent circuits of the two-port networks for various port conditions have been also presented. The use of these equivalent circuits allows the desired transmission properties to be obtained by means of the exact synthesis method.

Appendix

Derivation of the Design Formulas

It is desirable to derive the design formulas which are necessary for the determination of the cross-section dimensions from the given values of characteristic immittances by means of conformal mapping techniques. This can be done by transforming the boundary of the cross section into a simpler boundary for which the solution is known.

A. Cross Section of Fig. 8(b)

It is clear that the vertical centerline of the cross section shown in Fig. 8(b) is replaced by a magnetic wall for A and B-mode, or by an electric wall for C-mode, in consideration of the symmetry of the structure. Also, a magnetic wall can be placed along the horizontal centerline since the two conducting strips II-1 and II-2 are always at the same potential. Thus, we need consider in detail only one quarter of the complete cross section as shown in Fig. 11(a).

As a first step, the interior of the z-plane boundary in Fig. 11(a) is to be mapped into the first quadrant of the t-plane in Fig. 11(b). Electric walls are indicated by the solid lines and magnetic walls by the dotted lines. The dotted lines between D and E, and between G and O are for A and B-mode, while the solid lines are for C-mode. The letters denote pertinent points of the structure and will serve as references when transformations to different complex planes are made.

Transformation is carried out by the Schwarz-Christoffel method. The differential equation relating the z and t-plane is

$$\frac{dz}{dt} = C \frac{(\mathrm{sn}^2 \zeta - t^2)}{(1 - k^2 \cdot \mathrm{sn}^2 a \cdot t^2)\sqrt{(1 - t^2)(1 - k^2 t^2)}} \tag{38}$$

where C is a constant to be determined.

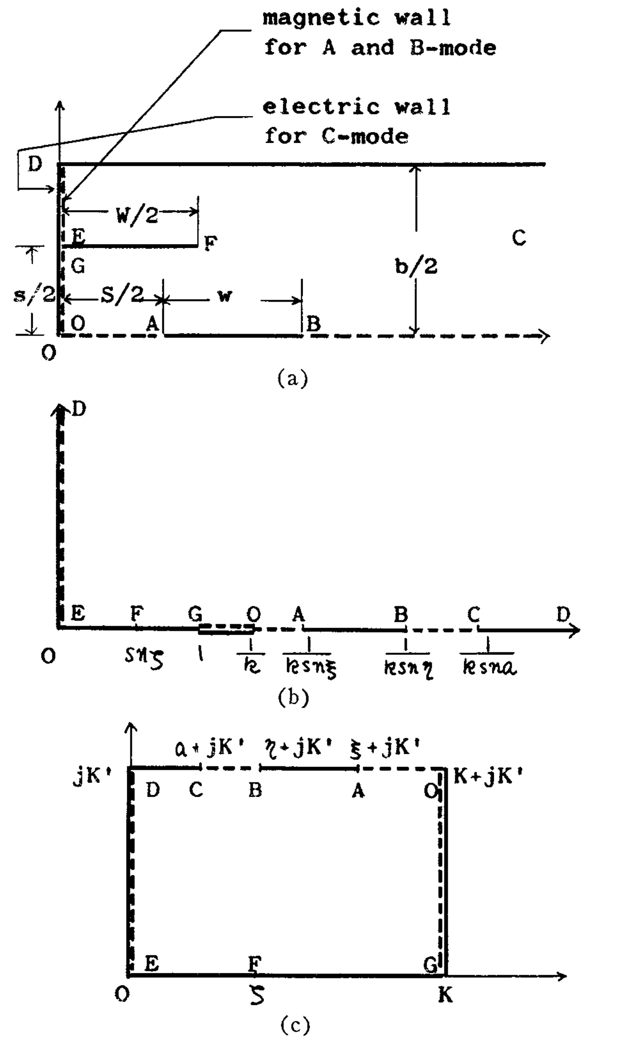


Fig. 11. Transformations used in the derivation of the design formulas for the cross section of Fig. 8(b). (a) z-plane. (b) t-plane. (c) u-plane.

Substituting

$$t = \mathrm{sn}\, u \tag{39}$$

in (38) yields the mapping function, after applying some boundary conditions,

$$z = \frac{b}{\pi} \left\{ u\mathrm{Z}(a) - \pi(u, a) \right\} + j\frac{s}{2} \tag{40}$$

under the following condition:

$$\mathrm{sn}^2 \zeta = \frac{1 - k^2 \,\mathrm{sn}^2 a \,\mathrm{sn}^2 \zeta}{k^2 \,\mathrm{sn}\, a \cdot \mathrm{cn}\, a \cdot \mathrm{dn}\, a} \mathrm{Z}(a). \tag{41}$$

In the above equations, sn u, cn u, and dn u are the Jacobian elliptic functions, $\mathrm{Z}(u)$ is the Jacobian zeta-function, and $\pi(u, a)$ is the elliptic integral of the third kind [19]. Equation (39) transforms the first quadrant of the t-plane to the interior of the fundamental rectangle of the Jacobian elliptic functions on the u-plane as shown in Fig. 11.

Next, solving (41) for ζ gives

$$\operatorname{sn}\zeta = \frac{1}{k}\cdot\frac{\sqrt{Z(a)}}{\sqrt{Z(a)}\operatorname{sn}^2 a + \operatorname{sn} a\cdot\operatorname{cn} a\cdot\operatorname{dn} a}\cdot \tag{42}$$

It now remains to relate the cross-sectional dimensions to the corresponding values of the u-plane. Using the Jacobian theta-function [19], $\Theta(u)$, (40) reduces to

$$z = \frac{b}{2\pi}\log\frac{\Theta(u+a)}{\Theta(u-a)} + j\frac{s}{2}\cdot \tag{43}$$

Applying the boundary conditions at F and O gives

$$\frac{W}{b} = \frac{1}{\pi}\log\frac{\Theta(\zeta+a)}{\Theta(\zeta-a)} \tag{44}$$

and

$$\frac{s}{b} = \frac{a}{K(k)} \tag{45}$$

where K is the complete elliptic integral of the first kind with modulus k.

Applying the boundary conditions at A and B gives

$$\frac{S}{b} = \frac{1}{\pi}\log\frac{H(\xi+a)}{H(\xi-a)} \tag{46}$$

and

$$\frac{w}{b} = \frac{1}{2\pi}\log\frac{H(\xi-a)\cdot H(\eta+a)}{H(\xi+a)\cdot H(\eta-a)}\cdot \tag{47}$$

In (46) and (47), the term H is the Jacobian eta-function [19] defined by

$$H(u) = -j\exp\left[j(2u+jK')\pi/4K\right]\Theta(u+jK') \tag{48}$$

where K' is the complete elliptic integral of the first kind with complementary modulus

$$k' = \sqrt{1-k^2}.$$

Thus, the normalized dimensions of the structure of Fig. 8(b) have been related to the u-plane parameters by (44) through (47).

1) A and B-Mode: Considering these modes, the first quadrant of the t-plane in Fig. 11(b) is mapped into the entire upper half of the t'-plane and finally into the infinite strip region of the χ-plane as shown in Fig. 12. These two mapping functions are obtained by the Schwarz-Christoffel method and are given by

$$t' = \frac{1}{1-k^2\operatorname{sn}^2 a\cdot t^2} \tag{49}$$

and

$$\chi = \frac{h}{2\pi}\log t'. \tag{50}$$

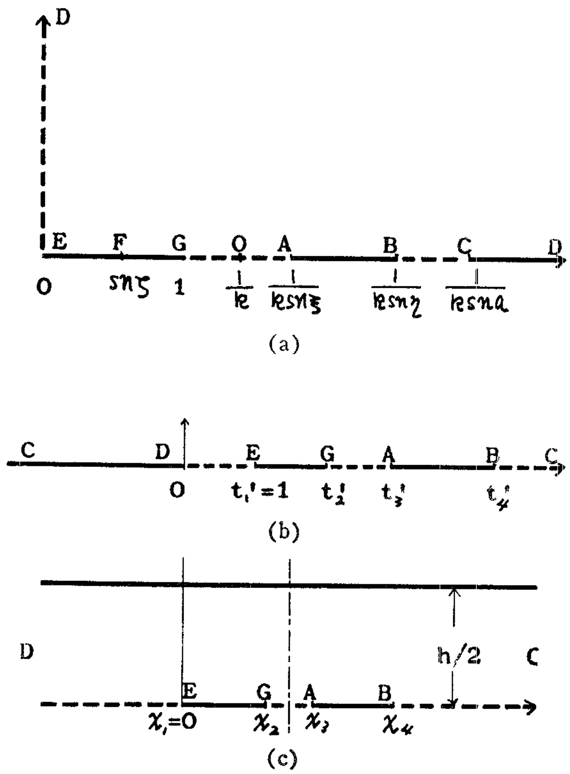


Fig. 12. Transformations for A and B-mode for the cross section of Fig. 8(b). (a) t-plane. (b) t'-plane. (c) χ-plane.

Inspection of the χ-plane in Fig. 12(c) shows that the field inside this bounded region is equivalent to that in the upper half cross section of the shielded coupled strip transmission line with two center conductors for which exact solution has been given by Cohn [20] for the symmetrical case. A-mode for the coupled strip transmission line treated here corresponds to even-mode for that with two center conductors, while B-mode corresponds to odd-mode. Furthermore, we find, in consideration of (4), that the χ-plane structure in Fig. 12(c) is symmetrical with respect to the vertical centerline, i.e.,

$$\chi_4 - \chi_3 = \chi_2 - \chi_1 = \chi_2 \tag{51}$$

where the χ's are the χ-plane dimensions.

Then characteristic immittances for A and B-mode are given by

$$\begin{aligned} Z_{oa} &= 2Z_{oA} = \frac{2}{Y_{oA}} = \frac{1}{Y_{oa}} = \frac{30\pi}{\sqrt{\epsilon_r}}\cdot\frac{K'(k_a)}{K(k_a)} \\ Z_{ob} &= 2Z_{oB} = \frac{2}{Y_{oB}} = \frac{1}{Y_{ob}} = \frac{30\pi}{\sqrt{\epsilon_r}}\cdot\frac{K'(k_b)}{K(k_b)} \end{aligned} \tag{52}$$

and the χ-plane dimensions are related to the moduli of the elliptic integrals by

$$\begin{aligned} \frac{\chi_2}{h} &= \frac{2}{\pi}\tanh^{-1}\lambda \\ \frac{\chi_3}{h} &= \frac{2}{\pi}\tanh^{-1}\mu \end{aligned} \tag{53}$$

where

$$\lambda = \sqrt{k_a k_b}$$
$$\mu = \sqrt{\frac{k_a}{k_b}}\,. \tag{54}$$

Now we will substitute corresponding values of the χ and t'-plane at G and A in (50) to get

$$\mathrm{sn}\; a = \frac{1}{k}\,\frac{2\sqrt{\lambda}}{1+\lambda}$$
$$\mathrm{sn}\; \xi = \frac{1+\mu}{2\sqrt{\mu}}\,\mathrm{sn}\; a. \tag{55}$$

Equation (51) can be written in terms of the t'-plane values as

$$t_4' = t_2' \cdot t_3'. \tag{56}$$

Substituting the t'-plane values at G, A, and B in (56) gives

$$\mathrm{sn}\; \eta = \frac{\mathrm{sn}\; a}{\sqrt{1 - \left(\frac{1-\mu}{1+\mu}\right)^2 \mathrm{dn}^2\, a}}\,. \tag{57}$$

2) C-Mode: Considering this mode, the first quadrant of the t-plane in Fig. 11(b) is mapped into the upper half of the t'-plane by

$$t' = \frac{\mathrm{sn}^2\, a - \mathrm{sn}^2\, \xi}{1 - \mathrm{sn}^2\, \xi} \cdot \frac{1 - k^2 t^2}{1 - k^2\, \mathrm{sn}^2\, a \cdot t^2} \tag{58}$$

and finally into the infinite strip region of the χ-plane by (50) as shown in Fig. 13. It can be seen that the field inside the χ-plane boundary in Fig. 13(c) is equivalent to that in the upper half of the shielded strip transmission line. Then characteristic immittances for C-mode are given by [21]

$$Z_{oc} = \frac{1}{Y_{oc}} = \frac{30\pi}{\sqrt{\epsilon_r}}\,\frac{K'(k_c)}{K(k_c)} \tag{59}$$

and strip width, χ_5, in Fig. 13(c) is related to k_c by

$$\frac{\chi_5}{h} = \frac{2}{\pi}\tanh^{-1} k_c. \tag{60}$$

Substitution of the corresponding values of the t and t'-plane at B in (58) yields, after simplification,

$$\frac{\mathrm{sn}^2\, \xi}{\mathrm{cn}^2\, \xi} \cdot \frac{\mathrm{cn}^2\, \eta}{\mathrm{dn}^2\, \eta} \cdot \frac{1}{\mathrm{dn}^2\, a} = \left(\frac{1+\nu}{1-\nu}\right)^2 \tag{61}$$

where

$$\nu = k_c. \tag{62}$$

Now letting

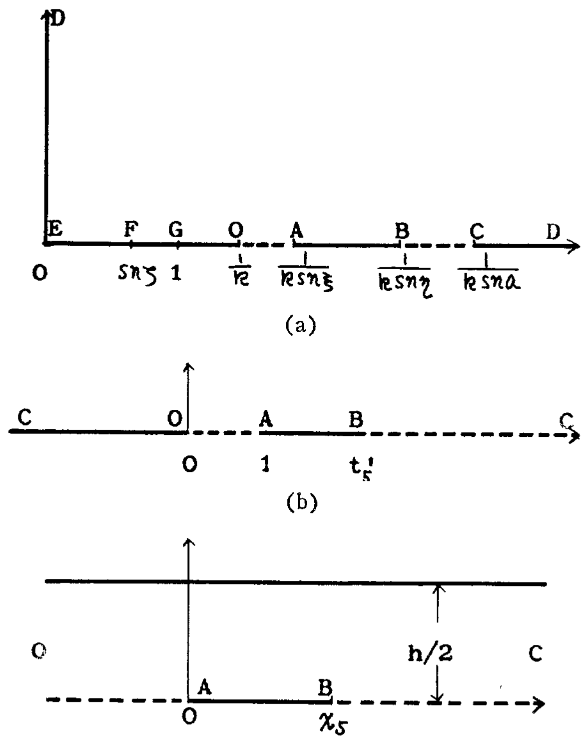


Fig. 13. Transformations for C-mode for the cross section of Fig. 8(b). (a) t-plane. (b) t'-plane. (c) χ-plane.

$$\lambda^* = \frac{1-\lambda}{1+\lambda}$$
$$\mu^* = \frac{1-\mu}{1+\mu}$$
$$\nu^* = \frac{1-\nu}{1+\nu} \tag{63}$$

and substituting (55) and (57) in (61), we obtain

$$k = \sqrt{\frac{(1-\lambda^{*2})(\lambda^{*2} - \nu^{*2})}{\lambda^{*2}(1-\mu^{*2}) - \nu^{*2}(1-\lambda^{*2}\mu^{*2})}}\,. \tag{64}$$

Thus the u-plane parameters in Fig. 11(c) have been related to the moduli of the elliptic integrals giving the characteristic immittances by (55), (57), and (64).

B. Cross Section of Fig. 8(c)

For A and B-mode, the procedure and mapping functions are the same for this case as for the previous case. Then we consider only C-mode, for which the vertical centerline of the cross section of Fig. 8(c) can be replaced by a magnetic wall and the horizontal centerline by an electric wall at zero potential. Therefore only one-quarter of the complete cross section is analyzed. The successive transformations for this case are shown in Fig. 14. Mapping functions are given by (39) and (40) under the condition of (41). Since the u-plane structure shown in Fig. 14(c) is the parallel-plate condenser with no fringing effect, characteristic immittances for C-mode are readily obtained as

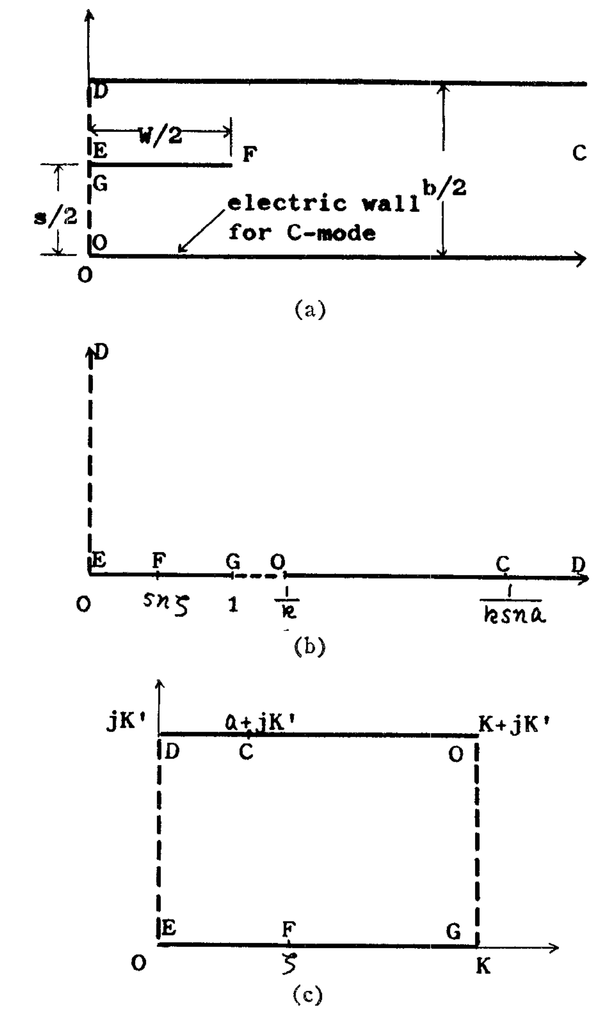


Fig. 14. Transformations for C-mode for the cross section of Fig. 8(c). (a) z-plane. (b) t-plane. (c) u-plane.

$$Z_{oc} = \frac{1}{Y_{oc}} = \frac{60\pi}{\sqrt{\epsilon_r}} \frac{K'(k_c)}{K(k_c)} \tag{65}$$

and the modulus is

$$k_c = k = \nu. \tag{66}$$

The design formulas are exactly the same for the cross section of Fig. 8(c) as for the cross section of Fig. 8(b), except that $Z_{oc}(Y_{oc})$ and k are given by (65) and (66) instead of (59) and (64).

References

[1] J. T. Bolljahn and G. L. Matthaei, "A study of the phase and filter properties of arrays of parallel conductors between ground planes," *Proc. IRE*, vol. 50, pp. 299–311, March 1962.
[2] H. Uchida, "Theory of multi-loop antennas," presented at the 1964 Antenna System Conf., Inst. of Elec. Commun. Engrs., Japan. (In Japanese.)
[3] N. Kuroyanagi, "Transmission line for high speed computers by using a tape cable," *J. Inst. Elec. Commun. Engrs. Japan*, vol. 45, pp. 1339–1347, October 1962. (In Japanese.)
[4] G. L. Matthaei, "Interdigital band-pass filters," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-10, pp. 479–491, November 1962.
[5] G. L. Matthaei, "Comb-line band-pass filters of narrow or moderate bandwidth," *Microwave J.*, vol. 6, pp. 82–91, August 1963.
[6] E. G. Cristal, "Coupled circular cylindrical rods between parallel ground planes," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-12, pp. 428–439, July 1964.
[7] J. W. Duncan, "Characteristic impedances of multiconductor strip transmission lines," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-13, pp. 107–118, January 1965.
[8] R. J. Wenzel, "Exact theory of interdigital band-pass filters and related coupled structures," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-13, pp. 559–575, September 1965.
[9] H. Ozaki and J. Ishii, "Synthesis of a class of strip-line filters," *IRE Trans. on Circuit Theory*, vol. CT-5, pp. 104–109, June 1958.
[10] R. J. Wenzel, "Exact design of TEM microwave networks using quarter-wave lines," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-12, pp. 94–111, January 1964.
[11] R. E. Collin, *Field Theory of Guided Waves*. New York: McGraw-Hill, 1960, p. 124.
[12] E. M. T. Jones and J. T. Bolljahn, "Coupled-strip-transmission-line filters and directional couplers," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-4, pp. 75–81, April 1956.
[13] K. Itakura, S. Yamamoto, and T. Azakami, "Coupled strip transmission line with three center conductors," presented at the 1965 Microwave Transmission System Conf., Inst. of Elec. Commun. Engrs., Japan. (In Japanese.)
[14] W. J. Getsinger, "A coupled strip-line configuration using printed-circuit construction that allows very close coupling," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-9, pp. 535–544, November 1961.
[15] K. Nagai, R. Sato, and N. Saito, "Coupled transmission-line filters," *1957 Proc. Joint Meeting of Inst. of Elec. Commun. Engrs.*, Japan, p. 328. (In Japanese.)
[16] A. Matsumoto, "Practical applications of multiwire networks," *1964 Proc. Microwaves, Circuit Theory, and Information Theory Conf.*, pt. 2, p. c-2-3.
[17] J. Ishii, "Characteristic impedances of coupled strip transmission line with unequal-width strips," *1959 Proc. Joint Meeting of Inst. of Elec. Commun. Engrs.*, Japan, p. 180. (In Japanese.)
[18] T. Ikeda, N. Saito, R. Sato, and K. Nagai, "Characteristic impedances of asymmetrical coupled microstrip line," *1963 Proc. Joint Meeting of Tohoku Branch of Inst. of Elec., of Elec. Commun., and of Illumin. Engrs.*, Japan, p. 3c–5. (In Japanese.)
[19] S. Tomochika, *Elliptic Functions*. Tokyo: Kyoritsu, 1958. (In Japanese.)
[20] S. B. Cohn, "Shielded coupled-strip transmission line," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-3, pp. 29–38, October 1955.
[21] S. B. Cohn, "Characteristic impedance of the shielded-strip transmission line," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-2, pp. 52–57, July 1954.